Math 429 - Exercise Sheet 2

1. Prove (at least for matrix groups) that the exponential

$$\mathfrak{g} \to G, \qquad X \to \exp(X)$$

is invariant under the adjoint action, i.e.

$$\exp(gXg^{-1}) = g\exp(X)g^{-1} \tag{1}$$

for any $X \in \mathfrak{gl}_n$ and $g \in GL_n$.

2. Then prove for any Lie group G that the abstractly defined exponential $\exp(X) = \gamma_X(1)$ satisfies

$$\exp(\mathrm{Ad}_g(X)) = \mathrm{Ad}_g(\exp(X)) \tag{2}$$

for any $g \in G$, $X \in \mathfrak{g}$.

3. Let $F : G \to G'$ be a Lie group homomorphism, and $f = F_* : \mathfrak{g} \to \mathfrak{g}'$ the induced linear map of tangent spaces. Show that we have

$$\boxed{\exp(f(X)) = F(\exp(X))} \tag{3}$$

for all $X \in \mathfrak{g}$. Conclude that f preserves the Lie bracket we defined last week

$$[X, X'] = \frac{\partial}{\partial t}\frac{\partial}{\partial t'} \exp(tX)\exp(t'X')\exp(tX)^{-1}\exp(t'X')^{-1}\Big|_{t=t'=0} \tag{4}$$

in the sense that

$$\boxed{f([X, X']) = [f(X), f(X')]} \tag{5}$$

for all $X, X' \in \mathfrak{g}$.

4. Consider the adjoint representaton $G \to GL(\mathfrak{g})$ and take its derivative

$$\mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \qquad \mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g} \tag{6}$$

for all $X \in \mathfrak{g}$. Then show that the Lie bracket(4) satisfies

$$[X, X'] = \mathrm{ad}_X(X') \tag{7}$$

for all $X, X' \in \mathfrak{g}$.

5. The following famous result of Baker-Campbell-Hausdorff shows how to reconstruct the multiplication in a Lie group G from the Lie bracket of $\mathrm{Lie}(G)$, at least in a neighborhood of the identity element.

Theorem 1. *If G is a Lie group, and $X, Y \in \mathrm{Lie}(G)$ are close enough to 0, then*

$$\exp(X)\exp(Y) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{a_1,\dots,a_n \geq 0 \\ b_1,\dots,b_n \geq 0}}^{a_i+b_i>0,\forall i} \frac{[X,\dots,[X,[Y,\dots,[Y,\dots,[X,\dots,[X,[Y,\dots,Y]\dots]}{(a_1+\dots+a_n+b_1+\dots+b_n)a_1!\dots a_n!b_1!\dots b_n!}\right) \tag{8}$$

where the inner sum involves the iterated Lie bracket of a_1 copies of X, followed by b_1 copies of Y, ..., followed by a_n copies of X, followed by b_n copies of Y.

Reverse engineer formula (8) as follows: suppose you're working in $G = GL_n$ and you want to find Z such that $\exp(X)\exp(Y) = \exp(Z)$, and Z is given by linear combinations of commutators of X and Y. Find the parts of Z which are linear, then quadratic, then cubic ... in X, Y (do so explicitly up to whatever order you can).

6. If the Baker-Campbell-Hausdorff formula was too much fun for you, then consider the following formula due to Campbell If G is a Lie group, and $X, Y \in \mathrm{Lie}(G)$ are close enough to 0, then

$$\mathrm{Ad}_{\exp(X)}(Y) = Y + [X,Y] + \frac{[X,[X,Y]]}{2!} + \frac{[X,[X,[X,Y]]]}{3!} + \dots$$

and prove it for $G = GL_n$.